Aug 2000

Day 1, Q1

Let $c_1, c_2, \dots, c_n$ be complex numbers. Define an $n \times n$ matrix $M$by

$$M_{ij} = c_i c_j$$

(a) Compute $\exp(M)$

Solution. Write the matrix M using a singular value decomposition. Notice, we can also write the matrix M as an outer product of complex numbers c CC^*. The Taylor expansion of M is then

$$\begin{aligned} M &= I + CC^* + \frac{(C,C)}{2}CC^* + \dots + \frac{(C,C)^n}{n!}CC^* + \dots \\ &= I + M(I + \frac{(C,C)}{2}I + \dots) \\ &= I + \left[\sum_{n=0}^{\infty} \frac{(C,C)^n}{n!}\right] M \\ &= I + e^{(C,C)} M \end{aligned}$$

♣

(b) Find the eigenvalues of M

Solution. Now for the eigenvalue problem. The problem reduces to

$$C(C^*C) = \lambda C$$

which has solution $\lambda = (C,C)$ of multiplicity 1, and $\lambda = 0$ of multiplicity $n-1$.

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Day 1, Q2

Evaluate

$$\oint \mathbf{F} \cdot d\mathbf{r},$$

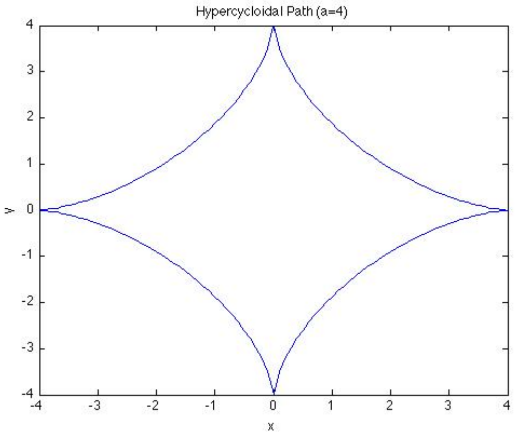
where $\mathbf{F} = (x^2 y \cos(x) + 2xy \sin(x) - y^2 e^x, x^2 \sin(x) - 2ye^x)$ and $d\mathbf{r} = (dx, dy)$, around the hypercycloidal path

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Solution. Consider $\mathbf{F} = P\hat{i} + Q\hat{j}$. Consider the partial derivatives $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$.

$$\begin{aligned} \frac{\partial P}{\partial y} &= x^2 \cos(x) + 2x \sin(x) - 2ye^x \\ \frac{\partial Q}{\partial x} &= 2x \sin(x) + x^2 \cos(x) - 2ye^x \end{aligned}$$

These two partial derivatives are equal, that is $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Therefore, the vector field is conservative. So, we know then that we can apply the fundamental thm. for line integrals, that is $\exists f(x, y)$ s.t. $\mathbf{F} = \nabla f$ and $\oint_C \mathbf{F} \cdot d\mathbf{r} = f(r_{\text{end}}) - f(r_{\text{start}})$. Let's take a look at the hypercycloidal curve.



This curve is closed. Therefore, we have a conservative vector field being integrated over a closed curve and so

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

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Day 1, Q3

Let A and B be 2×2 matrices such that

$$AB = 2BA$$

Show that all eigenvalues of the matrix AB equal zero.

Solution. First, make use of the relationship $\det(\alpha A) = \alpha^n \det(A)$ for A is an $n \times n$ matrix. For this problem, take $n = 2$, $\alpha = 2$.

Clearly, since $AB = 2BA$, the following is true

$$\det(AB) = \det(2BA) = 2^2 \det(BA) = 4 \det(AB).$$

Therefore, we know that $\det(AB) = 0$. Therefore, at least one eigenvalue must be zero. Since the characteristic equation for the matrix AB is simple, we can solve for both eigenvalues explicitly. The characteristic polynomial is

$$\lambda^2 - \text{Tr}(AB) + \det(AB) = 0$$

which has solutions

$$\lambda_{+/-} = \frac{1}{2}\big(\mathrm{Tr}(AB) \pm \sqrt{\mathrm{Tr}(AB) - 4\det(AB)}\big)$$

Clearly, we have $\lambda_- = 0$. For $\lambda_+ = \mathrm{Tr}(AB)$, we need that $\mathrm{Tr}(AB)$ is equal to 0. But we know that, since $AB = 2BA$,

$$\mathrm{Tr}(AB) = \mathrm{Tr}(2BA) = 2\mathrm{Tr}(BA) = 2\mathrm{Tr}(AB)$$

Therefore, $\mathrm{Tr}(AB) = 0$ and $\lambda_+ = 0$.

Therefore, both eigenvalues of AB are zero.

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Day 1, Q4

Show that $\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}dx = \frac{\pi}{4}(a+1)e^{-a}$ for $a > 0$.

Solution. Start with the definition of contour for use in integration. The function $f(z) = \frac{e^{ikz}}{(1+z^2)^2}$ has two poles (both of multiplicity 2). Pick the contour C to be the semicircle of radius $R \to \infty$ in the upper-half of the complex plane (i.e. we are enclosing the pole at $z = i$). Now, the integral we are interested in is

$$\frac{1}{2}\oint_C \frac{e^{iaz}}{(z-i)^2(z+i)^2}dz = \frac{1}{2}2\pi\mathrm{Res}_{z=i}(f(z)) \tag{1}$$

where $f(z) = \frac{e^{ikz}}{(1+z^2)^2}$.

The residue at $z = i$ is

$$\begin{aligned}
\mathrm{Res}_{z=i}(f(z)) &= \frac{1}{1!}\frac{d}{dz}(z-i)^2\frac{e^{iaz}}{(z-i)^2(z+i)^2}\bigg|_{z=i} \\
&= \frac{iae^{iaz}(z+i)^2 - e^{iaz}2(z+i)}{(z+i)^4}\bigg|_{z=i} \\
&= \frac{-4iae^{-a} - 4ie^{-a}}{16}
\end{aligned}$$

Multiply by πi to evaluate (1)

$$\begin{aligned}
\pi i\mathrm{Res}_{z=i}(f(z)) &= \pi i\frac{-4iae^{-a} - 4ie^{-a}}{16} \\
&= \frac{\pi}{4}(a+1)e^{-a}
\end{aligned}$$

So we have the relation

$$\begin{aligned}
\frac{1}{2}\oint_C \frac{e^{iaz}}{(z-i)^2(z+i)^2}dz &= \int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}dx + \oint_{\text{arc}} f(z)dz \\
&= \frac{\pi}{4}(a+1)e^{-a}
\end{aligned}$$

Now we just need that $\oint_{\text{arc}} f(z)dz = 0$. This is true through the application of Jordan's lemma.

Therefore

$$\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}dx = \frac{\pi}{4}(a+1)e^{-a}$$

♣

Day 1, Q5

Solve the heat equation $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$, $D > 0$ for $-\infty < x < \infty$ and $t > 0$, with the initial condition $u(x,0) = e^{-x^2}$.

Solution. Use separation of variables $u(x,t) = T(t)X(x)$ to separate the equation

$$\begin{aligned} XT' &= DTX'' \\ \Longrightarrow \frac{T'}{DT} &= \frac{X''}{X} = -\lambda \end{aligned}$$

where λ is a yet-to-be determined constant. This leads to the two equations

$$\begin{aligned} T' &= -\lambda DT \\ X'' &= -\lambda X \end{aligned}$$

The solution of the equations are

$$\begin{aligned} T &= B(\lambda)e^{-\lambda Dt} \\ X &= C(\lambda)\cos(\sqrt{\lambda}t) + D(\lambda)\sin(\sqrt{\lambda}t) \end{aligned}$$

We desire a solution that is not diverging in time, so we require $\lambda \geq 0$. Define $k^2 = \lambda$. Additionally, we know that our initial condition is an even function, therefore we only have a contribution from the cos functions. Therefore, reduce the coefficients to be of the form $A(k) = B(k)C(k)$.

The solutions then have the form

$$u(x,t;k) = A(k)e^{-k^2Dt}\cos(kx)$$

The full solution is the superposition (integral) of all such solutions $k > 0$, that is

$$\begin{aligned} u(x,t) &= \int_0^\infty A(k)e^{-k^2Dt}\cos(kx)dk \\ A(k) &= \frac{1}{\pi}\int_{-\infty}^\infty e^{-x^2}\cos(kx)dx \end{aligned}$$

where the $A(k)$ come from the initial condition $u(x,0) = e^{-x^2}$. We can make the claim about the $A(k)$ since e^{-x^2} is smooth over all of $\mathbb{R}$ and integrable, that is $||e^{-x^2}||_1 < \infty$. The $1/\pi$ is a normalization factor. Therefore, the final solution is

$$u(x,t) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^\infty e^{-x^2}\cos(kx)dx\right)\cos(kx)dk$$

Some further reduction is possible but seemingly unnecessary.

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Day 1, Q6

Find the sum of the series $\sum_{n=2}^{\infty} \frac{\sin^n(x)}{n(n-1)}$.

Solution. Notice that $|\sin^n x| \leq 1$. So, there is a good chance that we can use the geometric series relationship

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

in some way.

Differentiate the expression on the LHS to get (make the substitution $r = \sin x$)

$$\begin{aligned}
\frac{d^2}{dr^2} \sum_{n=2}^{\infty} \frac{r^n}{n(n-1)} &= \sum_{n=2}^{\infty} \frac{d^2}{dr^2} \frac{r^n}{n(n-1)} \\
&= \sum_{n=2}^{\infty} r^{n-2} \\
&= \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}
\end{aligned}$$

So, to find the desired sum, integrate twice. That is

$$\begin{aligned}
\int \frac{1}{1-r} dr &= \ln(1-r) \ \text{ integrate again (by parts)} \\
\int \ln(1-r) dr &= (1-r)(\ln(1-r)-1)
\end{aligned}$$

Substitute back $\sin x = r$

$$\sum_{n=2}^{\infty} \frac{\sin^n(x)}{n(n-1)} = (1-\sin x)\big(\ln(1-\sin x)-1\big)$$

and we're done.

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Day 1, Q7

Let $C_n \geq 0$ and define

$$f(x) = \sum_{n=0}^{\infty} C_n x^n$$

for $0 \leq x \leq 1$ ($f(x)$ may be infinite). Prove that

$$\int_0^{\infty} f(x) dx = \sum_{n=0}^{\infty} \frac{C_n}{n+1}$$

where the equality means either both sides are $+\infty$ or both sides are finite and they are equal.

Solution. Consider the sequence of partial sums $g_k(x) = \sum_{n=0}^{k} c_n x^n$. For $x \in (0,1)$, the sequence $g_k(x)$ is monotone increasing. So, by the MCT

$$\begin{aligned}
\int_0^1 f(x)dx &= \int_0^1 \lim_{k\to\infty} \sum_{n=0}^{k} C_n x^n dx \\
&= \lim_{k\to\infty} \int_0^1 \sum_{n=0}^{k} C_n x^n dx \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} C_n \int_0^1 x^n dx \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} \frac{C_n}{n+1} \\
&= \sum_{n=0}^{\infty} \frac{C_n}{n+1}
\end{aligned}$$

♣

Day 1, Q8

Consider the $n \times n$ "tri-diagonal" matrix

$$\begin{pmatrix}
c & a & 0 & 0 & 0 & \dots & b \\
b & c & a & 0 & 0 & \dots & 0 \\
0 & b & c & a & 0 & \dots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & 0 & b & c & a & 0 \\
0 & \dots & 0 & 0 & b & c & a \\
a & \dots & 0 & 0 & 0 & b & c
\end{pmatrix}$$

where a, b, and c are real constants. Find the eigenvectors and eigenvalues of this matrix.

Solution. This is a special matrix with well known eigenvalues/vectors. The eigenvalues are the n^{th} roots of unity

$$\lambda_m = e^{\frac{2\pi i m}{n}}$$

and eigenvectors

$$u^{(m)} = (1 \; e^{\frac{2\pi i m}{n}} \; \dots \; e^{\frac{2\pi i m(n-1)}{n}})^T$$

Day 1, Q9

Here is an initial value problem:

$$\begin{cases} y' = \lambda y, \quad \lambda \in \mathbb{C},\ t > 0 \\ y(0) = y_0. \end{cases}$$

Consider two cases for the value of the parameter λ: (a) $\lambda = -10^{-9}$ (b) $\lambda = i$. In case (a) the solution decays exponentially at a slow rate; in case (b) it is oscillatory with constant magnitude. The task is to analyze two numerical methods:

(1) The implicit Euler method $y_{n+1} = y_n + \Delta t \lambda y_{n+1}$. Take $\Delta t = 0.1$ and consider the limit $n \to \infty$. Discuss case (a). Discuss case (b). In each case compare the behavior of the numerical method with the behavior of the differential equation.

Solution. Find the relation of y_{n+1} in terms of $y(0) = y_0$

$$\begin{aligned} y_{n+1}(1-\lambda\Delta t) &= y_n && (2)\\ \implies y_{n+1} &= \frac{y_n}{1-\lambda\Delta t} && (3)\\ &= \frac{y_0}{(1-\lambda\Delta t)^n} && (4)\end{aligned}$$

For case (a), $\lambda = -10^{-9}$, $\Delta t = .1$. The above relation (4) becomes

$$y_{n+1} = \frac{y_0}{(1+10^{-10})^n}$$

which decays slowly in time (as $n \to \infty$). This is in agreement with the ODE and expected behavior of the solution.

For case (b), relation (4) becomes

$$\begin{aligned} y_{n+1} &= \frac{y_0}{(1-.1i)^n}\\ &= \frac{y_0(1+.1i)^n}{(1+.01)^n}\end{aligned}$$

which decays in time. This behavior is not indicative of the solution to the ODE, which does not decay in time.

We should conclude that method (1) works for case (a), but not so well for case (b).

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(2) The trapezoidal method $y_{n+1} = y_n + \frac{\Delta t}{2}\lambda(y_n + y_{n+1})$. Take $\Delta t = 0.1$ and consider the limit $n \to \infty$. Discuss case (a). Discuss case (b). In each case compare the behavior of the numerical method with the behavior of the differential equation.

Solution. The relationship for y_{n+1} in terms of y_0 is

$$y_{n+1} = \left(\frac{1+\frac{\lambda}{2}\Delta t}{1-\frac{\lambda}{2}\Delta t}\right)^n y_0 \qquad (5)$$

For case (a), relation (5) becomes

$$y_{n+1} = \left(\frac{1-5\cdot 10^{-11}}{1+5\cdot 10^{-11}}\right)^n y_0$$

which, again, decays very slowly. The decay is even slower than that in implicit Euler.

For case (b), relation (5) becomes

$$y_{n+1} = \left(\frac{1+\frac{i}{20}}{1-\frac{i}{20}}\right)^n y_0$$

The expression being raised to the n has mod 1. Therefore, there will be no decay in time. This behavior is commensurate with that of the analytic solution to the ODE.

♣

Day 1, Q10

The Green's function for

$$\begin{cases} u''(x) = f(x) & x \in [0,1] \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

is given by $G(x,y) = x(y-1)$ for $x < y$ and by $G(x,y) = (x-1)y$ for $x > y$. Construct the Green's function for

$$\begin{cases} u''(x) = f(x) & x \in [0,1] \\ u(0) = 0 \\ \int_0^1 u(x)dx = 0 \end{cases}$$

Solution. Start with the solution to the homogeneous eqn $u''(x) = 0$, which has solution $u(x) = Cx + D$, for some constants C, D. The Green's function of the ODE is

$$u(x;y) = \begin{cases} a(y)x + b(y) & 0 \le x \le y \\ c(y)x + d(y) & y \le x \le 1 \end{cases}$$

Apply the boundary condition $u(0) = 0$ to find that $b(y) = 0$. Apply continuity condition at $x = y$ to get the relationship

$$\begin{aligned} u_<(x;y) - u_>(x;y) &= 0 \\ a(y)y &= c(y)y + d(y) \\ \Longrightarrow d(y) &= \big(c(y) - a(y)\big)y \end{aligned}$$

Apply the jump condition at $x = y$ to get the relationship

$$\begin{aligned} \left.\frac{du_>}{dx} - \frac{du_<}{dx}\right|_{x=y} &= 1 \\ &= \big(c(y) - a(y)\big) \\ \Longrightarrow c(y) = \big(1 - a(y)\big) \end{aligned}$$

Now, finally, apply the second boundary condition to solve for $a(y)$.

$$\begin{aligned}
\int_0^1 u(x)dx &= \int_0^y u_<(x)dx + \int_y^1 u_>(x)dx \\
&= -\frac{3}{2}y^2 + y + \frac{1}{2}(a+1) = 0 \\
\implies a &= 3y^2 - 2y - 1
\end{aligned}$$

Now we use $a(y)$ to solve for the other coefficients. Combining all of this information gives

$$\begin{aligned}
a(y) &= 3y^2 - 2y - 1 \\
b(y) &= 0 \\
c(y) &= -3y^2 + 2y + 2 \\
d(y) &= y
\end{aligned}$$

Therefore, the solution is

$$u(x;y) = \begin{cases} (3y^2 - 2y - 1)x & 0 \le x \le y \\ (-3y^2 + 2y + 2)x + y & y < x \le 1 \end{cases}$$

♣

Day 1, Q11

Solve the nonlinear integral equation

$$y(x) = 1 + \frac{1}{4}\int_0^1 y^2(\xi)d\xi$$

Solution. The solution must be of the form $y(x) = C$ for some constant C since y is equal to a constant + its integral over a fixed interval. So, solve

$$\begin{aligned}
C &= 1 + \frac{1}{4}C^2(1-0) \\
&= 1 + \frac{1}{4}C^2 \\
\implies C &= 2
\end{aligned}$$

♣

Day 1, Q12

Let $C([0,1])$ denote the set of real valued, continuous functions on $[0,1]$. We define two norms on this set:

$$\begin{aligned}
||f||_1 &= \int_0^1 |f(x)|dx \\
||f||_\infty &= \sup_{x\in[0,1]} |f(x)|
\end{aligned}$$

Let $U = \{f \in C([0,1]) : f(x) > 0 \ \forall x\}$.

(a) Is U open in the $||\cdot||_1$ topology? Justify your answer.

Solution. To be open in the $|| \cdot ||_1$ topology means that for all elements $s \in S$ $\exists \epsilon$ s.t. $B_\epsilon(s) \subset S$.

The answer to the above question is false. We construct the necessary counterexample by finding a function g s.t. $||g - f||_1 \leq \epsilon$ but $g \notin U$.

Consider $f(x) = 1 > 0$. Clearly, $f(x) \in U$. Consider the function

$$g(x) \;=\; \begin{cases} \frac{4}{\epsilon}x - 1 & 0 \leq x \leq \frac{\epsilon}{2} \\ 1 & \frac{\epsilon}{2} < x \leq 1 \end{cases}$$

for any $\epsilon > 0$. The 1-norm reduces to

$$\begin{aligned} ||g - f||_1 &= \int_0^{\frac{\epsilon}{2}} 2 - \frac{4}{\epsilon}x dx \\ &= \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} < \epsilon \end{aligned}$$

However, $g(x) < 0$ for $x < \epsilon/4$, so $g \notin U$.

Therefore, U is not open in the $|| \cdot ||_1$ topology.

(b) Is U open in the $|| \cdot ||_\infty$ topology? Justify your answer.

Solution. The answer to the above question is yes. Consider an arbitrary $f \in U$ and consider the unit ball around f of radius ϵ, that is

$$B_\epsilon = \{g | ||g - f||_\infty < \epsilon\}$$

Let $m = \min_{x \in [0,1]} f(x)$. If $\epsilon < m$, then $||g - f||_\infty < \epsilon$ means $\min_{x \in [0,1]} g(x) > m - \epsilon > 0$. So, $g(x) > 0$ for all $x \in [0, 1]$.

Therefore, for all $f \in U$ there exists an $\epsilon > 0$ s.t. $B_\epsilon(f) \subset U$.

◆

Day 2, Q1

Find the leading order term in the asymptotic expansion of

$$\Gamma(k + 1) = \int_0^\infty e^{-t} t^k dt$$

as $k \to +\infty$.

Solution. Skipped.

◆

Day 2, Q2

Find a curve which is a critical point of the following functional

$$F(y) = \int_0^1 \left(\frac{1}{2}(y')^2 + yy' + y' \right) dx,\ y' = \frac{dy}{dx}.$$

when the end points $y(0)$ and $y(1)$ are **not** specified.

Solution. Consider the variation of $F(y)$, $\delta F = F(y + \delta y) - F(y)$.

$$\delta F \quad = \quad \left| \frac{\partial f}{\partial y'} \delta y \right|_0^1 + \int_0^1 \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] \delta y dx$$

where $f(x, y, y') = \frac{1}{2}(y')^2 + yy' + y'$.

A critical point is where $\delta F = 0$. To achieve this, first we require

$$\frac{\partial f}{\partial y} \quad = \quad \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right).$$

This gives us the differential equation we wish to solve. Plugging in the information for f, we have $y'' = 0$, or $y(x) = ax + b$ for some yet to be determined constants a, b. The second condition we need to exploit involves values on the boundary. Since we do not know the values of the end points, we require

$$\left. \frac{\partial f}{\partial y'} \right|_{x=0} \quad = \quad \left. \frac{\partial f}{\partial y'} \right|_{x=1} = 0.$$

Combine this information with $y(x) = ax + b$ to get the conditions on y

$$\begin{aligned} 2a + b + 1 &= 0 \\ a + b + 1 &= 0 \end{aligned}$$

so

$$\begin{aligned} a &= 0 \\ b &= -1 \end{aligned}$$

which gives us the solution

$$y(x) = -1$$

is a critical point of the functional F.

♣

Day 3, Q3

For $a > 0$, define

$$T_a(\varPhi) = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixna} \varPhi(x) dx$$

where $\varPhi(x)$ is in $\mathcal{S}$, the Schwartz space of test functions. Continuous linear functionals on $\mathcal{S}$ are called tempered distributions.

(a) Show that T_a is a tempered distribution.

Solution. Linearity is obvious; since the integral and the sum are both linear operations. We need only show continuity.

Start with the family of seminorms

$$q_{m,k}(\phi) = \max_{0 \le \alpha \le k} \sup_{x \in \mathbb{R}} |x|^m |\phi^{(\alpha)}(x)|$$

We would like to bound $|T_a(\phi)|$ by some combination of these seminorms that stays finite. This will show boundedness and thus continuity.

$$\begin{aligned}
T_a(\phi) &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixna} \phi(x) dx \\
&= \int_{-\infty}^{\infty} \phi(x) dx + \sum_{n \neq 0} \frac{1}{an} \int_{-\infty}^{\infty} e^{iu} \phi(\frac{u}{an}) du \\
&= \int_{-\infty}^{\infty} \frac{x^2 \phi(x)}{x^2} dx + \sum_{n \neq 0} \frac{1}{an} \left[\phi(\frac{u}{an}) \frac{1}{i} e^{iu} |_{-\infty}^{\infty} + i \int_{-\infty}^{\infty} e^{iu} \phi'(\frac{u}{an}) du \right]
\end{aligned}$$

We use the notion that $\phi \in \mathcal{S}$ means ϕ and its derivatives decay faster than any power of x. Look at the first integral:

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2 \phi(x)}{x^2} dx &= \int_{-\infty}^{-1} \frac{x^2 \phi(x)}{x^2} dx + \int_{-1}^{1} \phi(x) dx + \int_{1}^{\infty} \frac{x^2 \phi(x)}{x^2} dx \\
&\le 2q_{2,0}(\phi) + 2q_{0,0}(\phi)
\end{aligned}$$

The second integral becomes

$$\begin{aligned}
\sum_{n \neq 0} \frac{1}{an} \left[\phi(\frac{u}{an}) \frac{1}{i} e^{iu} |_{-\infty}^{\infty} + i \int_{-\infty}^{\infty} e^{iu} \phi'(\frac{u}{an}) du \right] &= \sum_{n \neq 0} \frac{i}{an} \int_{-\infty}^{\infty} e^{iu} \phi'(\frac{u}{an}) du \\
&= \sum_{n \neq 0} \frac{i}{an} \int_{-\infty}^{\infty} e^{iu} \frac{u^2 \phi'(\frac{u}{an})}{u^2} du \\
&\le \frac{1}{a^2} \sum_{n \neq 0} \frac{2q_{2,1}(\phi) + 2q_{0,1}(\phi)}{n^2}
\end{aligned}$$

The last series converges by the p-test. Therefore, we have

$$|T_a(\phi)| \leq 2(q_{2,0}(\phi) + q_{0,0}(\phi)) + \frac{2}{a^2}\sum_{n\neq 0}\frac{q_{2,1}(\phi) + q_{0,1}(\phi)}{n^2} < \infty$$

Therefore, $T_a(\phi)$ is continuous.

Therefore T_a is a tempered distribution.

♣

(b) Find $\lim_{a\to\infty} T_a$ in the sense of tempered distributions.

Solution. As $a \to \infty$, the sum in the above expression for T_a goes to zero. Therefore,

$$\lim_{a\to\infty} T_a(\phi) = \int_{-\infty}^{\infty} \phi(x)dx$$

This is the limit distribution.

♣

Day 2, Q4

Let $f(x)$ be a function on $\mathbb{R}$ and $\lambda > 0$. Define

$$f_\lambda(x) = f(\lambda x).$$

For what values of p ($1 \leq p \leq \infty$) is it true that

$$\lim_{\lambda\to 1} |f_\lambda - f|_p = 0, \quad \forall f \in L^p(\mathbb{R}, dx)$$

For each p you should either prove the statement is true or give a counterexample.

Solution. First, we need to make use of the following fact:

Definition. The set of smooth functions with compact support is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. That is, for some $\epsilon > 0$ there exists $\phi \in C_c^\infty(\mathbb{R})$ s.t. $||\phi - f||_p < \epsilon$ for all $f \in L^p(\mathbb{R})$.

Consider the p-norm

$$\begin{aligned} ||f - f_\lambda||_p &= ||(f-\phi) + (\phi - \phi_\lambda) + (\phi_\lambda - f_\lambda||_p \\ &\leq ||f-\phi||_p + ||\phi - \phi_\lambda||_p + ||\phi_\lambda - f_\lambda||_p \end{aligned}$$

Using the Definition, we can always find a smooth function ϕ close enough in L^p norm s.t. $||f - \phi||_p < \epsilon/2$ and $||\phi_\lambda - f_\lambda||_p < \lambda^{-1}\epsilon/2$. Now we need to look at the second integral

$$||\phi - \phi_\lambda||_p^p = \int_{-\infty}^{\infty} |\phi(x) - \phi(\lambda x)|^p dx$$

This integral is clearly bounded, since $|\phi(x) - \phi(\lambda x)| \leq 2\max_x |\phi|\mathbf{1}_{[-R,R]}$. So, by the DCT, we can move the limit inside the integral and clearly the integral goes to zero. Therefore,

$$\lim_{\lambda \to 1} ||f - f_\lambda||_p < \epsilon/2 + \epsilon/2 = \epsilon$$

For $p = \infty$, we don't have the density property from the Definition. In fact, the desired result no longer holds. A good counterexample is the Dirichlet function.

Day 2, Q5

In this problem we consider a discrete space variable that ranges over n equally spaced points $x_j = jh$ in the unit interval, where $h = 1/n$, and $j = 0, \ldots, n-1$. The space l^2 consists of real functions on these points with the inner product $(f, g) = \sum_{j=1}^{n-1} f(x_j)g(x_j)$. The difference operators are $D_+f(x) = (f(x+h) - f(x))/h$ and $D_-f(x) = (f(x) - f(x-h))/h$. In this context, we regard $x_{n-1} + h = x_0$ and $x_0 - h = x_{n-1}$. Note the adjoint relation $(D_+f, g) = -(f, D_-g)$.

In the problem the variable u depends on the discrete space variable and on continuous time $t \geq 0$. It is to compare stability properties of two schemes for differencing in the space variable. The central difference scheme is

$$\frac{du}{dt} + a\frac{1}{2}[D_+ + D_-]u = bD_+D_-u. \tag{6}$$

The scheme with the advection term treated with the upwind method is

$$\frac{du}{dt} + aD_-u = bD_+D_-u. \tag{7}$$

The parameters are taken so that the coefficient of the advection term is $a > 0$, while the coefficient of the diffusion term is $b \geq 0$.

(1) Find the parameter values for which the solution of (6) conserves l^2 norm as t increases. Find the parameter values for which it diminishes l^2 norm as t increases.

Solution. Multiply both sides of (6) by u and regroup terms

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(u)^2 + \frac{a}{2}(D_+u)u\frac{a}{2}(D_-u)u &= b(D_+D_-u)u \\
\implies \frac{d}{dt}\sum_{n=0}^{n-1} u^2 &= 2b\sum_{n=0}^{n-1}(D_+D_-u)u - a\sum_{n=0}^{n-1}(D_+u)u - a\sum_{n=0}^{n-1}(D_-u)u \\
\implies \frac{d}{dt}(u,u) &= 2b\sum_{n=0}^{n-1}(D_+D_-u)u - \frac{a}{2}(-(u, D_-u)) - \frac{a}{2}(D_-u, u) \\
&= 2b\sum_{n=0}^{n-1}(D_+D_-u)u + \frac{a}{2}(D_-u, u) - \frac{a}{2}(D_-u, u) \\
&= 2b(D_+D_-u, u)
\end{aligned}$$

If $b = 0$, then the 2-norm is unchanging in time. Otherwise, we would require constraints on u to make $(D_+D_-u, u) = 0$. Additionally, the 2-norm cannot be negative unless $(D_+D_-u, u) < 0$, as b cannot be negative by assumption.

♣

(2) Find the parameter values for which the solution of (7) conserves l^2 norm as t increases. Find the parameter values for which it diminishes l^2 norm as t increases.

Solution. Following the steps from part (1), we have the relation

$$\frac{d}{dt}(u,u) = 2b(D_+D_-u)u - 2a(D_-u,u)$$

If $b = \frac{a(D_-u,u)}{(D_+D_-u,u)}$, $\frac{d}{dt}(u,u) = 0$. If $b < \frac{a(D_-u,u)}{(D_+D_-u,u)}$, $\frac{d}{dt}(u,u) < 0$.

♣

Day 2, Q6

In the solution of $x(t) = 0$, $y(t) = 0$ of the nonlinear system

$$\begin{cases} \dot{x} = -x + y + xy \\ \dot{y} = x - y - x^2 - y^5 \end{cases}$$

globally stable? (Hint: Consider the distance from the origin.)

Solution. Take the hint. Consider

$$\begin{aligned} \frac{d}{dt}(x^2+y^2) &= 2x\dot{x} + 2y\dot{y} \\ &= 2x(-x+y+xy) + 2y(x-y-x^2-y^5) \\ &= -2x^2 + 2xy + 2yx - 2y^2 - 2yx^2 - 2y^6 \\ &= -2(x-y)^2 - 2y^6 \leq 0 \end{aligned}$$

All solutions to the system of ODEs will approach the $x(t) = y(t) = 0$. So, the solution is globally stable.

♣

Day 2, Q7

For $-1 < \mu < 0$ the following integral exists, and its value may be computed by contour integration to be

$$\int_0^\infty \frac{z^\mu}{1+z} dz = \frac{\pi}{\sin(-\pi\mu)}.$$

This fact is relevant to the following problem.

Consider the linear operator:

$$\varGamma\psi(x) = \int_0^\infty \frac{\psi(y)}{x+y} dy$$

defined on $L^1(\mathbb{R}^+, \frac{dx}{1+x})$.

1. Find eigenfunctions and eigenvalues of $\varGamma$.

Solution. Set up the eigenvalue problem for this operator

$$\begin{aligned}
\mathit{\Gamma}\phi(x) &= \lambda\phi(x) \\
\phi(x) &= \frac{1}{\lambda}\int_0^\infty \frac{\phi(x)}{x+y}dy \\
&= \frac{1}{\lambda x}\int_0^\infty \frac{\phi(y)}{1+\frac{y}{x}}dy \text{ change variables } u = y/x \\
&= \frac{1}{\lambda}\int_0^\infty \frac{\phi(xu)}{1+u}du
\end{aligned}$$

Take the hint, that is use eigenfunctions $\phi(xu) = x^\mu u^\mu$.

$$\begin{aligned}
x^\mu &= \frac{x^\mu}{\lambda}\int_0^\infty \frac{u^\mu}{1+u}du \\
&= \left(\frac{1}{\lambda}\frac{\pi}{\sin(-\pi\mu)}\right)x^\mu
\end{aligned}$$

so

$$\lambda = \frac{\pi}{\sin(-\pi\mu)}$$

Therefore, we have the eigenfunctions $\phi(x;\mu) = x^\mu$ with eigenvalues $\lambda_\mu = \frac{\pi}{\sin(-\pi\mu)}$.

♣

2. Is $\mathit{\Gamma}$ a bounded operator? Explain.

Solution. $\mathit{\Gamma}$ is unbounded. Consider the relation $||\mathit{\Gamma}||_1 \geq \sup_\mu\{\lambda_\mu\}$. For this problem, the eigenvalue spectrum is unbounded for μ any integer. Therefore $\sup_\mu\{\lambda_\mu\} = \infty$. Therefore, $\mathit{\Gamma}$ is an unbounded operator.

♣

Day 2, Q8

Let $f(x)$ be a periodic function on $[0, 2\pi]$ with the standard Fourier series

$$f(x) = \sum_n \big(a_n \cos(nx) + b_n \sin(nx)\big)$$

If $f(x)$ satisfies the Hölder condition

$$|f(x) - f(y)| \leq M|x-y|^\alpha \quad 0 \leq \alpha \leq 1$$

show that the Fourier coefficients satisfy

$$a_n \leq \frac{\pi^\alpha}{n^\alpha}M, \quad b_n \leq \frac{\pi^\alpha}{n^\alpha}M$$

Hint: Given that f is periodic, consider the properties of $f(x + \frac{\pi}{n})$.

Solution. Consider the relation for a_n,

$$\begin{aligned} a_n &= \int_{-\pi}^{\pi} f(x)\cos nx \frac{dx}{\pi} \\ &= \int_{-\pi}^{\pi} f(x+\pi/n)\cos(n(x+\pi/n))\frac{dx}{\pi} \\ &= \int_{-\pi}^{\pi} f(x+\pi/n)[\cos nx\cos\pi - \sin nx \sin\pi]\frac{dx}{\pi} \\ &= \int_{-\pi}^{\pi} f(x+\pi/n)\cos nx\cos\pi\frac{dx}{\pi} \end{aligned}$$

Add the two expressions and take the absolute value

$$\begin{aligned} 2|a_n| &\leq \int_{-\pi}^{\pi} |f(x+\pi/n)-f(x)||\cos nx|\frac{dx}{\pi} \text{ but } \cos nx \leq 1 \forall x \\ &\leq \frac{2\pi}{\pi}M\left(\frac{\pi}{n}\right)^{\alpha} \\ \implies |a_n| &\leq M\left(\frac{\pi}{n}\right)^{\alpha} \\ \implies a_n &\leq M\left(\frac{\pi}{n}\right)^{\alpha} \end{aligned}$$

We can follow the exact same steps as before using b_n instead of a_n to conclude

$$b_n \leq M\left(\frac{\pi}{n}\right)^{\alpha}$$